Local symmetry properties of graphs

Michael Giudici

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Automorphisms of graphs

Γ a finite simple connected graph.

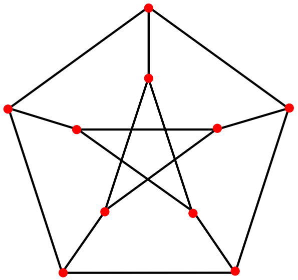
Unless otherwise stated, each vertex has valency at least 3.

Vertex set $V\Gamma$, edge set $E\Gamma$.

An automorphism of Γ is a permutation of the vertices which maps edges to edges.

$\mathrm{Aut}(\Gamma)$ is the group of all automorphisms of Γ.

Automorphisms of the Petersen graph



Rotations and reflections gives D_{10}.

Interchange inside with outside.

This gives 20 automorphisms.

$\mathrm{Aut}(\Gamma) = S_5$

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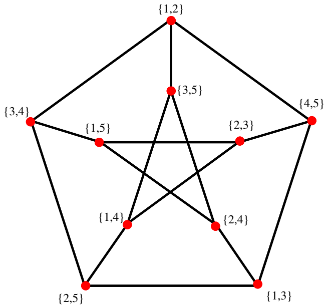

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$\text{Aut}(\Gamma) = S_5$

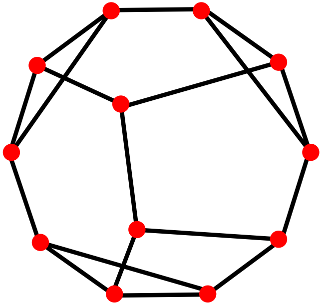
Vertex-transitive graphs

Say Γ is vertex-transitive if $\mathrm{Aut}(\Gamma)$ acts transitively on $V\Gamma$, that is, for any two vertices v and w there is an automorphism g mapping v to w.

The Petersen graph is vertex-transitive.

Such graphs are regular, for g induces a bijection from $\Gamma(v)$ to $\Gamma(w)$.

Frucht graph



The Frucht graph is regular but has trivial automorphism group.

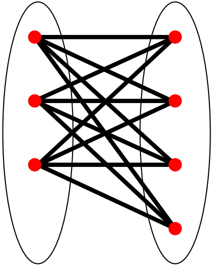
Edge-transitive graphs

Say Γ is edge-transitive if $\mathrm{Aut}(\Gamma)$ acts transitively on $E\Gamma$.

The Petersen graph is edge-transitive

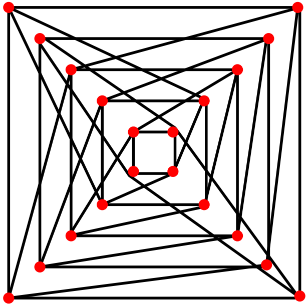
Suppose that Γ is edge-transitive but not vertex-transitive.

Then each vertex-orbit contains a unique vertex from each edge



Thus only two orbits of vertices and these are the two biparts.

Folkman graph



Edge-transitive, regular but not vertex-transitive.

Arc-transitive

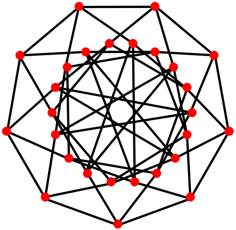
We say Γ is arc-transitive if $\mathrm{Aut}(\Gamma)$ acts transitively on the set $A\Gamma$ of arcs, that is on all ordered pairs of adjacent vertices.

The Petersen graph is arc-transitive.

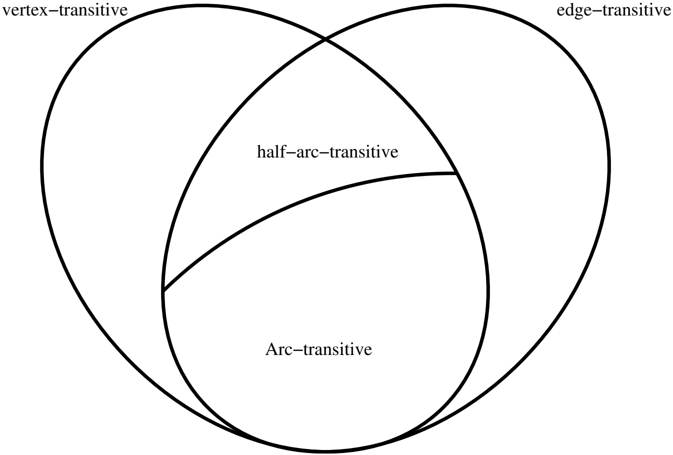
Arc-transitive implies edge-transitive and vertex-transitive.

Vertex- and edge-transitive but not arc-transitive graphs are called half-arc-transitive.

Holt graph



Interaction



Coset graphs

- G a group with subgroup H,
- $g \in G \backslash H$ such that $g^2 \in H$.

We can construct the graph $Cos(G, H, HgH)$ with

vertex set: cosets of H in G
adjacency: $Hx \sim Hy$ if and only if $xy^{-1} \in HgH$

G acts by right multiplication on vertices and is transitive on $A\Gamma$.

Any arc-transitive graph Γ can be constructed in this way:

- $G = \mathrm{Aut}(\Gamma)$, $H = G_v$
- g an element interchanging v and w, where $\{v, w\} \in E\Gamma$.

Petersen graph: $G = S_5$, $H = G_{\{1,2\}}$ and $g = (13)(24)$.

Coset graphs II

- a group G with subgroups L and R

We can construct the bipartite graph $Cos(G, L, R)$ with

vertex set:	cosets of L in G and cosets of R in G
adjacency:	$Lx \sim Ry$ if and only if $Lx \cap Ry \neq \varnothing$ or equivalently, if $xy^{-1} \in LR$

G acts by right multiplication with two orbits on vertices and transitive on $E\Gamma$.

Any edge-transitive bipartite graph can be constructed in this way: $G = \mathrm{Aut}(\Gamma)$, $L = G_v$ and $R = G_w$ for some edge $\{v, w\}$.

s-arc transitive graphs

An s-arc in a graph is an $(s+1)$-tuple $(v_0, v_1, \ldots, v_s)$ of vertices such that $v_i \sim v_{i+1}$ and $v_{i-1} \neq v_{i+1}$.

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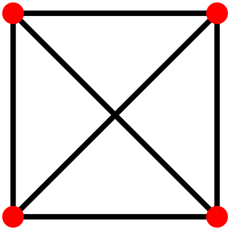

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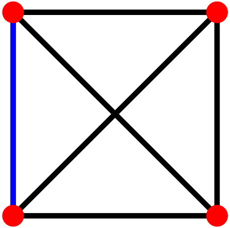

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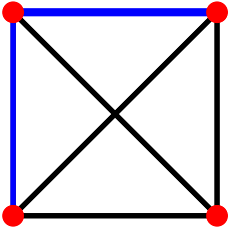

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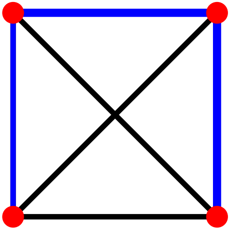

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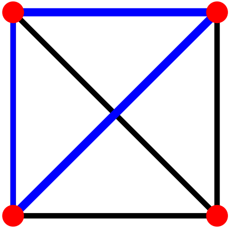

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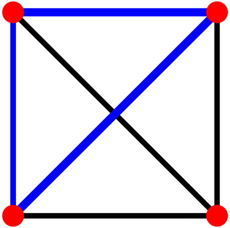

A graph is s-arc transitive if $\mathrm{Aut}(\Gamma)$ is transitive on the set of s-arcs.

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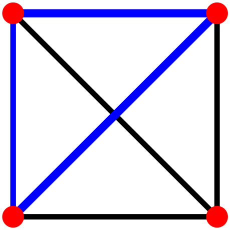
An s-arc in a graph is an $(s+1)$-tuple $(v_0, v_1, \ldots, v_s)$ of vertices such that $v_i \sim v_{i+1}$ and $v_{i-1} \neq v_{i+1}$.



A graph is s-arc transitive if $\mathrm{Aut}(\Gamma)$ is transitive on the set of s-arcs.

K_4 is 2-arc transitive but not 3-arc transitive.

Some basic facts

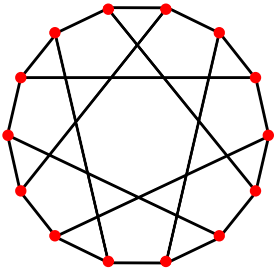
s-arc transitive implies $(s-1)$-arc transitive.

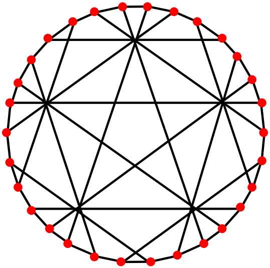
In particular, s-arc transitive implies arc-transitive and hence vertex-transitive.

If $G \leqslant \mathrm{Aut}(\Gamma)$ such that G acts transitively on s-arcs we say that Γ is (G, s)-arc transitive.

Examples

- Cycles are s-arc transitive for arbitrary s.
- Complete graphs are 2-arc transitive.
- Petersen graph is 3-arc transitive.
- Heawood graph (point-line incidence graph of Fano plane) is 4-arc transitive.
- Tutte-Coxeter graph (point-line incidence graph of the generalised quadrangle $W(3, 2)$) is 5-arc transitive.





Bounds on s

Tutte (1947,1959): For cubic graphs, $s \leq 5$.

Weiss (1981): For valency at least 3, $s \leq 7$.

Upper bound is met by the generalised hexagons associated with $G_2(q)$ for $q = 3^n$.

These are bipartite, with valency $q + 1$ and $2(q^5 + q^4 + q^3 + q^2 + q + 1)$ vertices.

Local action

Γ is $(G, 2)$-arc transitive if and only if G_v is 2-transitive on $\Gamma(v)$ and G transitive on $V\Gamma$.

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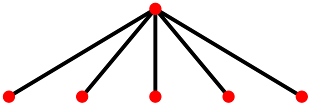

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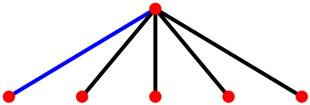

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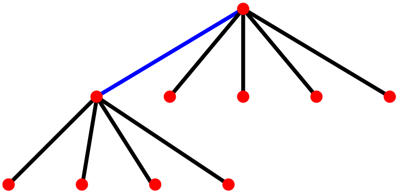


Structure of vertex stabiliser

Tutte: For a cubic graph which is s-arc transitive but not $(s+1)$-arc transitive, $|G_v| = 3.2^{s-1}$.

Djoković and Miller (1980): Determined the possible structures of a vertex stabiliser in cubic case: only 7 possibilities.

Use knowledge of 2-transitive groups to study possible vertex stabilisers.

Quotient graphs

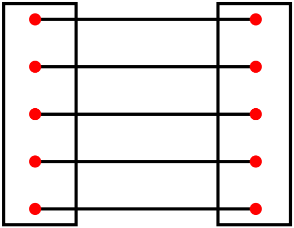
$\mathcal{B}$ a partition of $V\Gamma$

Quotient graph $\Gamma_{\mathcal{B}}$:

vertex set: parts of $\mathcal{B}$

adjacency: $B_1 \sim B_2$ if there exists $v_1 \in B_1$ and $v_2 \in B_2$ such that $v_1 \sim v_2$.

Γ is a cover of $\Gamma_{\mathcal{B}}$ if:



Quotients of s-arc transitive graphs

The quotient of a 2-arc transitive graph is not necessarily 2-arc transitive.

Babai (1985): Every finite regular graph has a 2-arc transitive cover.

Normal quotients and basic graphs

Instead look at normal quotients, that is, where $\mathcal{B}$ is the set of orbits of some normal subgroup N of $G \leqslant \mathrm{Aut}(\Gamma)$.

Denote by Γ_N.

Theorem (Praeger 1993)

Let Γ be a (G, s)-arc transitive graph and $N \lhd G$ with at least three orbits on $V\Gamma$. Then Γ_N is (G, s)-arc transitive. Moreover, Γ is a cover of Γ_N.

So the basic (G, s)-arc transitive graphs to study are those for which all nontrivial normal subgroups of G have at most two orbits.

Quasiprimitive groups

A permutation group is quasiprimitive if every nontrivial normal subgroup is transitive.

Praeger (1993) proved an O'Nan-Scott Theorem for quasiprimitive groups which classifies them into 8 types.

Only 4 are possible for a 2-arc transitive group of automorphisms.

- Affine: Ivanov-Praeger (1993), 2^d vertices.
- Twisted Wreath: Baddeley (1993)
- Product Action: Li-Seress (2006+)
- Almost Simple:

Li (2001): 3-arc transitive implies Almost Simple or Product Action.

Bipartite case

Let Γ be a bipartite graph with group G acting transitively on $V\Gamma$.

G has an index 2 subgroup G^+ which fixes the two halves setwise.

In particular, G cannot be quasiprimitive.

The basic graphs to study are those where every normal subgroup of G has at most two orbits, ie G is biquasiprimitive on vertices.

Structure theory of biquasiprimitive groups given by Praeger (2003).

In fact G^+ may or may not be quasiprimitive on each orbit.

See Alice Devillers' talk.

Locally s-arc transitive

In the bipartite graph case, the index two subgroup G^+ contains each vertex stabiliser G_v.

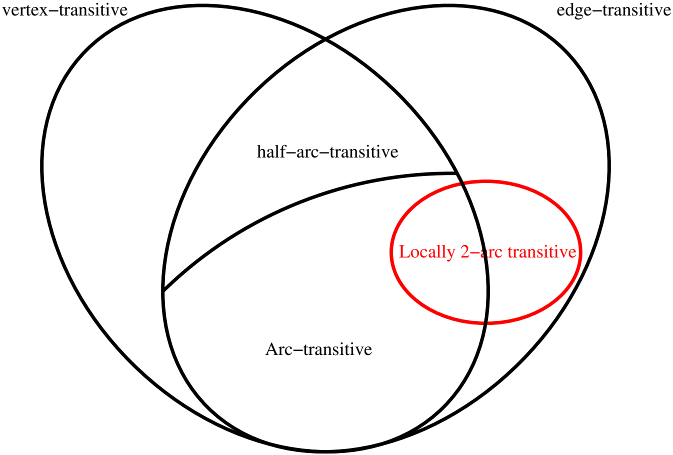
In particular, $(G^+)_v = G_v$ and so $(G^+)_v$ acts transitively on the set of all s-arcs starting at v.

We say that Γ is locally (G, s)-arc transitive if for all vertices v, G_v acts transitively on the set of s-arcs starting at v.

- G_v is 2-transitive on $\Gamma(v)$.
- If G is transitive on vertices then Γ is (G, s)-arc-transitive.
- If G is intransitive on vertices then G has two orbits and Γ is bipartite.

eg point-line incidence graph of a projective space

Interaction



Bounds on s

Stellmacher (1996): $s \leq 9$

Bound attained by classical generalised octagons associated with ${}^2F_4(q)$ for $q = 2^n$, n odd.

These have valencies $\{2^n + 1, 2^{2n} + 1\}$.

Main approach of study has been to determine possibilities for $G_v^{\Gamma(v)}$ and $G_w^{\Gamma(w)}$ for some edge $\{v, w\}$ and try to determine $\{G, G_v, G_w\}$.

Global approach

Theorem (Giudici-Li-Praeger (2004))

- Γ a locally (G, s)-arc transitive graph,
- G has two orbits Δ_1, Δ_2 on vertices,
- $N \lhd G$.

1. If N intransitive on both Δ_1 and Δ_2 then Γ_N is locally $(G/N, s)$-arc transitive. Moreover, Γ is a cover of Γ_N.
2. If N transitive on Δ_1 and intransitive on Δ_2 then Γ_N is a star.

Basic graphs

There are two types of basic locally (G, s)-arc transitive graphs:

(i) G acts faithfully and quasiprimitively on both Δ_1 and Δ_2.

(ii) G acts faithfully on both Δ_1 and Δ_2 and quasiprimitively on only Δ_1. (The star case)

Theorem (Giudici-Li-Praeger (2004))

1. In case (i), either
   - the quasiprimitive types of G^{Δ_1} and G^{Δ_2} are the same and one of 4 possibilities, or
   - one is Simple Diagonal while the other is Product Action.
2. In case (ii) there are only 5 possibilities for the type of G^{Δ_1}.

The $\{\mathrm{SD}, \mathrm{PA}\}$ case

All characterised by Giudici-Li-Praeger (2006-07).

Either $s \leq 3$ or the following locally 5-arc transitive example:

$\Gamma = Cos(G, L, R)$ with

- $G = \mathrm{PSL}(2, 2^m)^{2^m} \rtimes \mathrm{AGL}(1, 2^m)$, $m \geq 2$,
- $L = \{(t, \dots, t) \mid t \in \mathrm{PSL}(2, 2^m)\} \times \mathrm{AGL}(1, 2^m)$,
- $R = (C_2^{2m} \rtimes C_{2^m-1}) \rtimes \mathrm{AGL}(1, 2^m)$

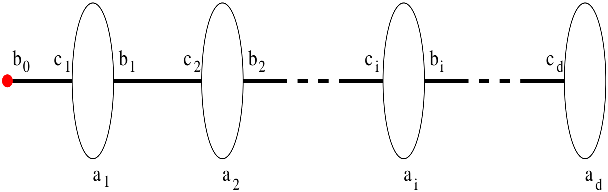
On the set of cosets of R, G preserves a partition into $(2^m + 1)^{2^m}$ parts.

- valencies $\{2^m + 1, 2^m\}$
- $G_v^{\Gamma(v)} = \mathrm{PSL}(2, 2^m)$, $G_w^{\Gamma(w)} = \mathrm{AGL}(1, 2^m)$

Important place in the Stellmacher/van Bon program.

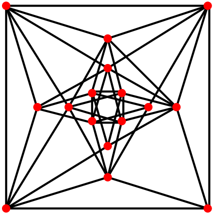
Distance transitive graphs

Γ is called distance transitive if for each i, $\mathrm{Aut}(\Gamma)$ is transitive on the set $\{(v, w) \mid d(v, w) = i\}$.



A graph satisfying these regularity properties is called distance regular.

Shrikhande graph



Imprimitive distance transitive graphs

Smith (1971)

An imprimitive distance transitive graph is either bipartite or antipodal.

In bipartite case, the distance two graph $\Gamma^{(2)}$ has two connected components, each distance-transitive.

In the antipodal case, the antipodal quotient is distance-transitive.

Primitive distance transitive graphs

Praeger-Saxl-Yokoyama (1987): A primitive distance transitive graph

- can be derived from a Hamming graph, or
- is of Almost Simple or Affine type.

Classification is almost complete.

Locally distance transitive graphs

Say Γ is locally distance transitive if for each vertex v and integer i, $\mathrm{Aut}(\Gamma)_v$ acts transitively on the set of vertices at distance i from v.

- If Γ is vertex-transitive then it is distance transitive.
- If Γ is not vertex-transitive then $\mathrm{Aut}(\Gamma)$ has two orbits on vertices and Γ is bipartite.

The distance parameters for a vertex only depend on the part of the bipartition it belongs to.

eg line-plane incidence graph of a projective space.

Locally distance transitive graphs II

Shawe-Taylor (1987)

- If Γ is locally distance transitive and bipartite then $\Gamma^{(2)}$ has two connected components, each of which is distance transitive.
- In the nonregular case, at least one is primitive.

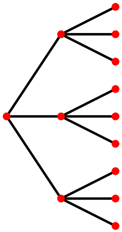
So use knowledge of primitive distance transitive graphs.

Locally s-distance transitive and s-distance transitive

Joint work with Alice Devillers, Cai Heng Li, Cheryl Praeger

Γ is called locally (G, s)-distance transitive if $s \leq \mathrm{diam}(\Gamma)$, and for each vertex v and $i \leq s$, G_v acts transitively on $\Gamma_i(v)$.

A (G, s)-distance transitive graph is a locally (G, s)-distance transitive graph such that G is transitive on $V\Gamma$.



If $s \leq \lfloor \frac{g-1}{2} \rfloor$, where g is the length of the shortest cycle, then Γ is (locally) s-distance transitive if and only if Γ is (locally) s-arc transitive.

Quotienting?

In bipartite case, the connected components of $\Gamma^{(2)}$ have half the diameter of Γ.

Paths in Γ may decrease in length in Γ_N and indeed Γ_N may have smaller diameter than Γ.

Quotienting

Let $LDT(s)$ be the set of graphs Γ that are locally s'-distance transitive where $s' = \min\{s, \mathrm{diam}(\Gamma)\}$.

Theorem (Devillers-Giudici-Li-Praeger)

Let $s \geq 2$ and let $\Gamma \in LDT(s)$ relative to G and let $N \lhd G$ with at least three orbits on vertices. Then one of the following holds:

- $\Gamma = K_{m[b]}$,
- Γ_N *is a star,*
- $\Gamma_N \in LDT(s)$ *relative to* G/N *and* Γ *is a cover of* Γ_N.

Basic graphs

There are four types of basic locally (G, s)-distance transitive graphs to study:

- G acts quasiprimitively on $V\Gamma$;
- Γ is bipartite, G is biquasiprimitive on $V\Gamma$ and G^+ acts faithfully on each orbit;
- Γ is bipartite, $G = G^+$ acts faithfully and quasiprimitively on each orbit;
- Γ is bipartite, $G = G^+$ acts faithfully on both orbits and quasiprimitively on only one.

These are currently under investigation.